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Discrete Mathematics 308 (2008) 2600–2606

DISCRETE MATHEMATICS

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Note

Weighted Bartholdi zeta functions of graph coverings ☆

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Received 14 January 2004; received in revised form 17 July 2006; accepted 19 July 2006
Available online 10 May 2007

Abstract

We define the weighted Bartholdi zeta function and a weighted L-function of a graph G, and give determinant expressions of them. Furthermore, we present a decomposition formula for the weighted Bartholdi zeta function of a regular covering of G by weighted L-functions of G.

Keywords: Zeta function; Graph; Cycle; Bump

1. Introduction

Graphs and digraphs treated here are finite. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $R(G) = \{(u, v), (v, u) | uv \in E(G)\}$ be the set of oriented edges (or arcs) (u, v), (v, u) directed oppositely for each edge uv of G. For $e = (u, v) \in R(G)$, $u = o(e)$ and $v = t(e)$ is called the *origin* and the *terminal* of e, respectively. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path* P *of length* n in G is a sequence $P = (e_1, \dots, e_n)$ of n arcs such that $e_i \in R(G)$, $t(e_i) = o(e_{i+1}) (1 \leqslant i \leqslant n-1)$. If $e_i = (v_{i-1}, v_i)$, $1 \leqslant i \leqslant n$, then we also denote $P = (v_0, v_1, \dots, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$*-path*. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \dots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \dots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \dots, e_m)$ and $C_2 = (f_1, \dots, f_m)$ are called *equivalent* if there exists an integer k such that $f_j = e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C.

We say that a path $P = (e_1, \dots, e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e_{i+1}^{-1} = e_i$ for some $i (1 \leqslant i \leqslant n-1)$. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if C^2 has no bump. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly shorter cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

☆ Supported by Grant-in-Aid for Science Research (C).
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0012-365X/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.disc.2006.07.044

The (*Ihara*) *zeta function* of a graph G is defined to be a function of a complex variable t with $|t|$ sufficiently small, by

$$\mathbf{Z}(G,t)=\mathbf{Z}_G(t)=\prod_{[C]}(1-t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C (see [9]).

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [9]. In [9], he showed that their reciprocals are explicit polynomials. Hashimoto [8] treated multivariable zeta functions of bipartite graphs. Bass [3] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial:

$$\mathbf{Z}(G,t)^{-1}=(1-t^2)^{r-1}\det(\mathbf{I}-t\mathbf{A}(G)+t^2(\mathbf{D}-\mathbf{I})),$$

where r and $\mathbf{A}(G)$ is the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D}=(d_{ij})$ is the diagonal matrix with $d_{ii}=\deg v_i$ and $d_{ij}=0, i\neq j\,(V(G)=\{v_1,\dots,v_n\})$.

Stark and Terras [14] gave an elementary proof of Bass' Theorem, and discussed three different zeta functions of a general graph. Furthermore, various proofs of Bass' Theorem were given by Foata and Zeilberger [5], and Kotani and Sunada [10].

Cycles, reduced cycles and prime cycles in a digraph are defined in the same manner as the case of a graph. Let D be a connected digraph. Then, the *zeta function* of D is defined to be a function of a complex variable t with $|t|$ sufficiently small by

$$\mathbf{Z}(D,t)=\mathbf{Z}_D(t)=\prod_{[C]}(1-t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D. The zeta function of a digraph is much easier to handle than that of a graph. For example, it is well-known (cf, [4]) and easy to check

$$\mathbf{Z}(D,t)^{-1}=\det(\mathbf{I}-\mathbf{A}(D)t),$$

where $\mathbf{A}=\mathbf{A}(D)$ is the adjacency matrix of D. Kotani and Sunada [10] stated that the zeta function of a finite graph is equal to that of its oriented line graph, and gave a simple proof of Bass' Theorem.

Let G be a connected graph. Then the *cyclic bump count* $cbc(C)$ of a cycle $C=(e_1,\dots,e_n)$ is

$$cbc(C)=|\{i=1,\dots,n\,|\,e_i=e_{i+1}^{-1}\}|,$$

where $e_{n+1}=e_1$. Then the *Bartholdi zeta function* of G is defined to be a function of complex variables u,t with $|u|,|t|$ sufficiently small by

$$\zeta_G(u,t)=\zeta(G,u,t)=\prod_{[C]}(1-u^{cbc(C)}t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G (see [2]). If $u=0$, then the Bartholdi zeta function of G is the (Ihara) zeta function of G.

Let n and m be the number of vertices and unoriented edges of G, respectively. Then two $2m\times 2m$ matrices $\mathbf{B}=(\mathbf{B}_{e,f})_{e,f\in R(G)}$ and $\mathbf{J}=(\mathbf{J}_{e,f})_{e,f\in R(G)}$ are defined as follows:

$$\mathbf{B}_{e,f}=\begin{cases}1 & \text{if } t(e)=o(f),\\ 0 & \text{otherwise},\end{cases}\qquad \mathbf{J}_{e,f}=\begin{cases}1 & \text{if } f=e^{-1},\\ 0 & \text{otherwise}.\end{cases}$$

Bartholdi [2] extended a result by Grigorchuk [6] relating cogrowth and spectral radius of random walks, and gave an explicit formula determining the number of bumps on paths in a graph. Furthermore, he presented the "circuit series" of the free products and direct products of graphs, and obtained a generalized form of the Ihara(-Selberg) zeta function.

Theorem 1 (*Bartholdi*). *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\zeta(G,u,t)^{-1}=\det(\mathbf{I}_{2m}-(\mathbf{B}-(1-u)\mathbf{J})t)=(1-(1-u)^2t^2)^{m-n}\det(\mathbf{I}-t\mathbf{A}(G)+(1-u)(\mathbf{D}-(1-u)\mathbf{I})t^2).$$

In the case of $u=0$, Theorem 1 implies Bass' Theorem. If $u=1$, then Theorem 1 gives a determinant expression of the zeta function of the symmetric digraph D corresponding to a graph G.

Foata and Zeilberger [5] gave a new proof of Bass' Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ a total order in X, and X^* the free monoid generated by X. Then the total order $<$ on X derive the lexicographic order $<$ on X^*. A *Lyndon word* in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l for any $r \geqslant 2$, and which is also minimal in the class of its cyclic rearrangements under $<$(see [11]). Let L denote the set of all Lyndon words in X.

Foata and Zeilberger [5] gave a short proof of Amitsur's identity [1].

Theorem 2 (*Amitsur*). *For square matrices* $\mathbf{A}_1, \dots, \mathbf{A}_k$,

$$\det(\mathbf{I} - (\mathbf{A}_1 + \cdots + \mathbf{A}_k)) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1, \dots, k\}$, *and* $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ *for* $l = i_1 \cdots i_p$.

In Section 2, we define the weighted Bartholdi zeta function of a graph G, and give determinant expressions of it. In Section 3, we give a decomposition formula for the weighted Bartholdi zeta function of a regular covering of G. In Section 4, we define a weighted L-function of G, and present a determinant expression of it. As a corollary, we present a decomposition formula for the weighted Bartholdi zeta function of a regular covering of G by weighted L-functions of G.

For a general theory of the representation of groups, the reader is referred to [13].

2. Weighted Bartholdi zeta functions of graphs

Let G be a connected graph and $V(G) = \{v_1, \dots, v_n\}$. We associate with each of its arc $e = (v_i, v_j)$ a complex variable $w_e = w(v_i, v_j)$. For each path $P = (v_{i_1}, \dots, v_{i_r})$ of G, the *weight* $w(P)$ of P is defined as follows: $w(P) = w(v_{i_1}, v_{i_2}) w(v_{i_2}, v_{i_3}) \dots w(v_{i_{r-1}}, v_{i_r})$. The *weighted Bartholdi zeta function* of G is defined by

$$\zeta(G, w, u) = \prod_{[C]} (1 - u^{cbc(C)} w(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G.

Let n and m be the number of vertices and unoriented edges of G, respectively. Then two $2m \times 2m$ matrices $\mathbf{W} = (\mathbf{W}_{e,f})_{e,f \in R(G)}$ and $\mathbf{T} = (\mathbf{T}_{e,f})_{e,f \in R(G)}$ are defined as follows:

$$\mathbf{W}_{e,f} = \begin{cases} w(e) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{T}_{e,f} = \begin{cases} w(e) & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. *Let G be a connected graph. Then the reciprocal of the weighted Bartholdi zeta function of G is given by*

$$\zeta(G, w, u)^{-1} = \det(\mathbf{I} - (\mathbf{W} - (1-u)\mathbf{T})).$$

Proof. Let $R(G) = \{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$. For each arc $e_r \in R(G)$, let $\mathbf{X}_{e_r}$ be the $2m \times 2m$ matrix whose the r row is the r row of $\mathbf{W} - (1-u)\mathbf{T}$, and whose other rows are $\mathbf{0}$. Set $\mathbf{M} = \mathbf{I} - \sum_{e \in R(G)} \mathbf{X}_e$. Then, for any sequence of arcs π,

$$\det(\mathbf{I} - \mathbf{X}_\pi) = \begin{cases} \det(\mathbf{I} - u^{cbc(\pi)} w(\pi)) & \text{if } \pi \text{ is a cycle,} \\ 1 & \text{otherwise.} \end{cases}$$

By Theorem 2, we have

$$\zeta(G, w, u)^{-1} = \det \mathbf{M}. \qquad \square$$

3. Weighted Bartholdi zeta functions of regular coverings

Let G be a connected graph, and let $N(v)=\{w \in V(G) | (v,w) \in R(G)\}$ for any vertex v in G. A graph H is called a *covering* of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the *quotient graph* G/Π is a simple graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group Aut H of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : R(G) \longrightarrow \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v,u)=\alpha(u,v)^{-1}$ for each $(u,v) \in R(G)$. The pair (G,α) is called an *ordinary voltage graph*. The *derived graph* G^{α} of the ordinary voltage graph (G,α) is defined as follows:

$$V(G^{\alpha})=V(G)\times\Gamma \text{ and } ((u,h),(v,k)) \in R(G^{\alpha}) \text{ if and only if } (u,v) \in R(G) \text{ and } k=h\alpha(u,v).$$

The *natural projection* $\pi : G^{\alpha} \longrightarrow G$ is defined by $\pi(u,h)=u$, $(u,h) \in V(G^{\alpha})$. The graph G^{α} is called a *derived graph covering* of G with voltages in Γ or a *Γ-covering* of G. The natural projection π commutes with the right multiplication action of the $\alpha(e)$, $e \in R(G)$ and the left action of $g \in \Gamma$ on the fibers: $g \circ (u,h)=(u,gh)$, $g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^{α} is a $|\Gamma|$-fold regular covering of G with covering transformation group Γ. Furthermore, every regular covering of a graph G is a Γ-covering of G for some group Γ(see [7]).

Let G be a connected graph, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. In the Γ-covering G^{α}, set $v_g=(v,g)$ and $e_g=(e,g)$, where $v \in V(G)$, $e \in R(G)$, $g \in \Gamma$. For $e=(u,v) \in R(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1}=(e^{-1})_{g\alpha(e)}$.

Let $w : R(G) \longrightarrow \mathbf{C}$ be a weight of G. Then we define the *weight* $\tilde{w}$ *of* G^{α} *derived from* w as follows:

$$\tilde{w}(u_g,v_h) := \begin{cases} w(u,v) & \text{if } (u,v) \in R(G) \text{ and } h=g\alpha(u,v), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, two matrices $\tilde{\mathbf{W}}=\mathbf{W}(G^{\alpha})=(\tilde{w}(e_g,f_h))$ of G^{α} and $\tilde{\mathbf{T}}=\mathbf{T}(G^{\alpha})=(\tilde{t}(e_g,f_h))$ of G^{α} are given by

$$\tilde{w}(e_g,f_h) := \begin{cases} w(e) & \text{if } t(e_g)=o(f_h), \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{t}(e_g,f_h) := \begin{cases} w(e) & \text{if } e_g^{-1}=f_h, \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, let the matrix $\tilde{\mathbf{W}}_g=(w_{ef}^{(g)})$ be defined by

$$w_{ef}^{(g)} := \begin{cases} w(e) & \text{if } \alpha(e)=g \text{ and } t(e)=o(f), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let the matrix $\tilde{\mathbf{T}}_g=(t_{ef}^{(g)})$ be defined by

$$t_{ef}^{(g)} := \begin{cases} w(e) & \text{if } \alpha(e)=g \text{ and } e^{-1}=f, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1,\ldots,\mathbf{M}_s$. If $\mathbf{M}_1=\mathbf{M}_2=\cdots=\mathbf{M}_s=\mathbf{M}$, then we write $s \circ \mathbf{M}=\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 4. *Let G be a connected graph with l unoriented edges, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let $\rho_1=1,\rho_2,\ldots,\rho_t$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1=1$. Suppose that the Γ-covering G^{α} of G is connected. Then the reciprocal of the weighted Bartholdi zeta function of G^{α} is*

$$\zeta(G^{\alpha},\tilde{w},u)^{-1}=\zeta(G,w,u)^{-1}\cdot\prod_{i=2}^{t}\det\left(\mathbf{I}_{2lf_i}-\sum_{h\in\Gamma}\rho_i(h)\otimes(\tilde{\mathbf{W}}_h-(1-u)\tilde{\mathbf{T}}_h)\right)^{f_i}.$$

Proof. Let $R(G) = \{e_1, \dots, e_l, e_{l+1}, \dots, e_{2l}\}$ and $\Gamma = \{1 = g_1, g_2, \dots, g_m\}$. Arrange arcs of G^α in m blocks: $(e_1, 1), \dots, (e_{2l}, 1)$; $(e_1, g_2), \dots, (e_{2l}, g_2)$; $\cdots$; $(e_1, g_m), \dots, (e_{2l}, g_m)$. We consider the matrix $\tilde{\mathbf{W}} - (1-u)\tilde{\mathbf{T}}$ under this order. For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $t(e, g_i) = o(f, g_j)$ if and only if $t(e) = \mathrm{o}(f)$ and $(\mathrm{o}(f), g_j) = \mathrm{o}(f, g_j) = t(e, g_i) = (t(e), g_i\alpha(e))$, i.e., $\alpha(e) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus we have

$$\tilde{\mathbf{W}} - (1-u)\tilde{\mathbf{T}} = \sum_{h \in \Gamma} \mathbf{P}_h \otimes (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h).$$

Let ρ be the right regular representation of Γ. Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_t$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_t \circ \rho_t(h)$ for each $h \in \Gamma$ (see [13]). Putting $\mathbf{B} = (\mathbf{P}^{-1} \otimes \mathbf{I}_{2l})(\tilde{\mathbf{W}} - (1-u)\tilde{\mathbf{T}})(\mathbf{P} \otimes \mathbf{I}_{2l})$, we have

$$\mathbf{B} = \sum_{h \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_t \circ \rho_t(h)\} \otimes (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h).$$

Note that $\mathbf{W} - (1-u)\mathbf{T} = \sum_{h \in \Gamma} (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h)$ and $1 + f_2^2 + \cdots + f_t^2 = m$. Therefore it follows that

$$\begin{aligned}\zeta(G^\alpha, \tilde{w}, u)^{-1} &= \det(\mathbf{I}_{2lm} - (\tilde{\mathbf{W}} - (1-u)\tilde{\mathbf{T}})) \\ &= \det(\mathbf{I}_{2l} - (\mathbf{W} - (1-u)\mathbf{T})) \prod_{i=2}^{t} \det\left(\mathbf{I}_{2lf_i} - \sum_h \rho_i(h) \otimes (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h)\right)^{f_i}. \qquad \square\end{aligned}$$

4. Weighted L-function of graphs

Let G be a connected graph, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ and d its degree. The *weighted L-function* of G associated to ρ and α is defined by

$$\zeta_G(w, u, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - \rho(\alpha(C))u^{cbc(C)}w(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G.

Let $1 \leqslant i, j \leqslant n$. Then, the *$(i, j)$-block* $\mathbf{B}_{i,j}$ of an $dn \times dn$ matrix $\mathbf{B}$ is the submatrix of $\mathbf{B}$ consisting of $d(i-1)+1, \dots, di$ rows and $d(j-1)+1, \dots, dj$ columns.

Theorem 5. *Let G be a connected graph with l unoriented edges, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ, and d the degree of ρ. Then the reciprocal of the weighted L-function of G associated to ρ and α is*

$$\zeta_G(w, u, \rho, \alpha)^{-1} = \det\left(\mathbf{I} - u \sum_{h \in \Gamma} \rho(h) \otimes (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h)\right).$$

Proof. At first, let $R(G) = \{e_1, \dots, e_l, e_{l+1}, \dots, e_{2l}\}$ and consider the lexicographic order on $R(G) \times R(G)$ derived from a total order of $R(G)$: $e_1 < e_2 < \cdots < e_{2l}$. If (e_i, e_j) is the mth pair under the above order, then we define the $2ld \times 2ld$ matrix $\mathbf{W}_m = ((\mathbf{W}_m)_{p,q})_{1 \leqslant p,q \leqslant 2l}$ as follows:

$$(\mathbf{W}_m)_{p,q} = \begin{cases} \rho(\alpha(e_i))w(e_i) & \text{if } p = i, q = j \text{ and } t(e_i) = \mathrm{o}(e_j), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Furthermore, we define the $2ld \times 2ld$ matrix $\mathbf{T}_m = ((\mathbf{T}_m)_{p,q})_{1 \leqslant p,q \leqslant 2l}$ as follows:

$$(\mathbf{T}_m)_{p,q} = \begin{cases} \rho(\alpha(e_i))w(e_i) & \text{if } p = i, q = j \text{ and } e_i^{-1} = e_j, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let $\mathbf{B} = (\mathbf{W}_1 - (1-u)\mathbf{T}_1) + \cdots + (\mathbf{W}_k - (1-u)\mathbf{T}_k)$, $k = 4l^2$. Then we have

$$\mathbf{B} = \sum_h (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h) \otimes \rho(h).$$

Let L be the set of all Lyndon words in $R(G) \times R(G)$. Then we can also consider L as the set of all Lyndon words in $\{1, \ldots, k\}$: $(e_{i_1}, e_{j_1}) \cdots (e_{i_q}, e_{j_q})$ corresponds to $m_1 m_2 \cdots m_q$, where $(e_{i_r}, e_{j_r})(1 \leqslant r \leqslant q)$ is the m_rth pair. Theorem 2 implies that

$$\det(\mathbf{I}_{2ld} - \mathbf{B}) = \prod_{t \in L} \det(\mathbf{I}_{2ld} - (\mathbf{W}_t - (1-u)\mathbf{T}_t)),$$

where

$$\mathbf{W}_t - (1-u)\mathbf{T}_t = (\mathbf{W}_{i_1} - (1-u)\mathbf{T}_{i_1}) \cdots (\mathbf{W}_{i_p} - (1-u)\mathbf{T}_{i_p})$$

for $t = i_1 \cdots i_p$. Note that $\det(\mathbf{I}_{2ld} - (\mathbf{W}_t - (1-u)\mathbf{T}_t))$ is the alternating sum of the diagonal minors of $\mathbf{W}_t - (1-u)\mathbf{T}_t$. Thus, we have

$$\det(\mathbf{I} - (\mathbf{W}_t - (1-u)\mathbf{T}_t)) = \begin{cases} \det(\mathbf{I} - \rho(\alpha(C))u^{cbc(C)}w(C)) & \text{if } t \text{ is a prime cycle } C, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$\begin{aligned} \zeta_G(w, u, \rho, \alpha)^{-1} &= \det\left(\mathbf{I}_{2ld} - \sum_{h \in \Gamma} (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h) \otimes \rho(h)\right) \\ &= \det\left(\mathbf{I}_{2ld} - \sum_{h \in \Gamma} \rho(h) \otimes (\tilde{\mathbf{W}}_h - (1-u)\tilde{\mathbf{T}}_h)\right). \qquad \square \end{aligned}$$

By Theorems 4, 5, the following result holds.

Corollary 1. *Let G be a connected graph, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\zeta(G^\alpha, \tilde{w}, u) = \prod_\sigma \zeta_G(w, u, \sigma, \alpha)^{\deg \sigma},$$

where σ runs over all inequivalent irreducible representations of Γ.

Let G be a connected graph, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be any representation of Γ and $d = \deg \rho$. The *L-function* of G associated to ρ and α is defined by

$$\zeta_G(u, t, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - \rho(\alpha(C))u^{cbc(C)}t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G.

Let $w_{ij} = t$ unless $w_{ij} = 0$. Then we obtain Corollary 1 in [12].

Corollary 2 (*Mizuno and Sato*). *Let G be a connected graph, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Suppose that the Γ-covering G^α of G is connected. Then we have*

$$\zeta(G^\alpha, u, t) = \prod_\rho \zeta_G(u, t, \rho, \alpha)^d,$$

where ρ runs over all inequivalent irreducible representations of Γ and $d = \deg \rho$.

Proof. At first, we have

$$\zeta(G^{\alpha}, \tilde{w}, u) = \zeta(G^{\alpha}, u, t),$$

and

$$\zeta_G(w, u, \rho, \alpha) = \zeta_G(u, t, \rho, \alpha).$$

By Corollary 1, the result follows. □

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